Performance Metrics

Ricardo Rocha and Fernando Silva

Computer Science Department

Faculty of Sciences

University of Porto

Parallel Computing 2023/2024

Performance

Two key goals to be achieved with the design of parallel applications are:

- **Performance** – the capacity to reduce the time needed to solve a problem as the computing resources increase
- **Scalability** – the capacity to maintain or increase performance as the size of the problem increases

The main factors limiting the performance and the scalability of an application can be divided into:

- **Architectural limitations**
- **Algorithmic limitations**

Factors Limiting Performance

Architectural limitations:

- **Latency and bandwidth**
- **Data coherency**
- **Memory capacity**

Algorithmic limitations:

- **Missing parallelism** (sequential code)
- **Communication frequency**
- **Synchronization frequency**
- **Poor scheduling** (task granularity/load balancing)

Performance Metrics for Parallel Applications

Some of the best known metrics are:

- ***Speedup***
- **Efficiency**

There also some laws/metrics that try to explain and assert the potential performance of a parallel application. The best known are:

- **Amdahl's law**
- **Gustafson-Barsis' law**
- **Karp-Flatt metric**
- **Isoefficiency metric**

Speedup

Speedup is a measure of performance. It measures the ratio between the sequential execution time and the parallel execution time.

$$S(p) = \frac{T(1)}{T(p)}$$

T(1) is the execution time with one processing unit

T(p) is the execution time with ***p*** processing units

	1 CPU	2 CPUs	4 CPUs	8 CPUs	16 CPUs
$T(p)$	1000	520	280	160	100
$S(p)$	1	1.92	3.57	6.25	10.00

Efficiency

Efficiency is a measure of the usage of the computational capacity. It measures the ratio between performance and the number of resources available to achieve that performance.

$$E(p) = \frac{S(p)}{p} = \frac{T(1)}{p \times T(p)}$$

S(p) is the speedup for ***p*** processing units

	1 CPU	2 CPUs	4 CPUs	8 CPUs	16 CPUs
$S(p)$	1	1.92	3.57	6.25	10.00
$E(p)$	1	0.96	0.89	0.78	0.63

Sublinear and Linear Speedup

We say that the speedup is **sublinear** when the ratio between the sequential execution time and the parallel execution time with ***p*** processing units is smaller than ***p***. This is the (most common) case where the **parallel execution shows some (significant) communication costs**.

$$\frac{T(1)}{T(p)} < p \Leftrightarrow E(p) < 1$$

We say that the speedup is **linear** (perfect scenario) when the ratio between the sequential execution time and the parallel execution time with ***p*** processing units is close or equal to ***p***. In this scenario, the **parallel execution shows negligible or no communication costs**.

$$\frac{T(1)}{T(p)} \approx p \Leftrightarrow E(p) \approx 1$$

Superlinear Speedup

We say that the speedup is **superlinear** (ideal scenario) when the ratio between the sequential execution time and the parallel execution time with ***p*** processing units is greater than ***p***.

$$\frac{T(1)}{T(p)} \geq p \Leftrightarrow E(p) > 1$$

Some factors that may make the speedup superlinear are:

- Almost inexistent initialization, comunication and/or synchronization costs
- Increased memory capacity (the problem may start to fit all in memory)
- Subdivision of the problem (smaller tasks may generate less cache misses)
- Computation randomness in optimization problems or with multiple solutions

Amdahl's Law

We can divide the computations performed by a parallel application in three major classes:

- ***C(seq)*** – computations that can be done only sequencially
- ***C(par)*** – computations that can be done in parallel
- ***C(com)*** – computations related to parallel communication and synchronization

Using these classes, the speedup of an application can be defined as:

$$S(p) = \frac{T(1)}{T(p)} = \frac{C(seq) + C(par)}{C(seq) + \frac{C(par)}{p} + C(com)}$$

Amdahl's Law

Since $C(com) \geq 0$ then:

$$S(p) \leq \frac{C(seq)+C(par)}{C(seq)+\frac{C(par)}{p}}$$

Let $f \in [0,1]$ be the **fraction of the computation that can be realized only sequentially**:

$$f = \frac{C(seq)}{C(seq)+C(par)} \qquad and \qquad S(p) \leq \frac{\frac{C(seq)}{f}}{C(seq)+\frac{C(seq)\times\left(\frac{1}{f}-1\right)}{p}}$$

Amdahl's Law

Simplifying:

$$S(p) \le \frac{\frac{C(seq)}{f}}{C(seq) + \frac{C(seq) \times \left(\frac{1}{f} - 1\right)}{p}}$$

$$\Rightarrow \quad S(p) \le \frac{\frac{1}{f}}{1 + \frac{\frac{1}{f} - 1}{p}} \quad \Rightarrow \quad S(p) \le \frac{1}{f + \frac{1-f}{p}}$$

Amdahl's Law

Let ***f*** $\in$ **[0,1]** be the fraction of the computation that can be realized only sequentially then Amdahl's law tells us that the maximum speedup that a parallel application can achieve with ***p*** processing units is:

$$S(p) \le \frac{1}{f + \frac{1-f}{p}}$$

Amdahl's law can also be used to determine the **limit of maximum speedup that a parallel application can achieve** regardless of the number of processing units available.

$$\lim_{p \to \infty} \frac{1}{f + \frac{1-f}{p}} = \frac{1}{f} \quad \Rightarrow \quad S(p) \le \frac{1}{f}$$

Applying Amdahl's Law

Through experimentation, it was verified that a given program spends 90% of the execution time in computation that can be parallelizable. Using Amdahl's law, estimate the maximum speedup that can be achieved with a parallel version of the application executing on 8 and 16 processing units?

$$S(8) \leq \frac{1}{0.1 + \frac{1-0.1}{8}} \approx 4.71 \quad \text{and} \quad S(16) \leq \frac{1}{0.1 + \frac{1-0.1}{16}} \approx 6.40$$

And with an arbitrary number of processing units?

$$\lim_{p \to \infty} \frac{1}{0.1 + \frac{1-0.1}{p}} = 10$$

Limitations of Amdahl's Law

Amdahl's law **ignores the costs with communication/synchronization operations** associated with the parallel version of the problem. For that reason, **it may provide a too optimistic upper bound for the speedup**.

		1 CPU	2 CPUs	4 CPUs	8 CPUs	16 CPUs
Amdahl's law	n = 10000	1	1.95	3.70	6.72	11.36
	n = 20000	1	1.98	3.89	7.51	14.02
	n = 30000	1	1.99	3.94	7.71	14.82
Speedup measure	n = 10000	1	1.61	2.11	2.22	2.57
	n = 20000	1	1.87	3.21	4.71	6.64
	n = 30000	1	1.93	3.55	5.89	9.29

Gustafson-Barsis' Law

Consider again the speedup measure defined previously:

$$S(p) \leq \frac{C(seq) + C(par)}{C(seq) + \frac{C(par)}{p}}$$

Let ***f*** be the **fraction of the parallel computation spent executing sequential computations** then ***(1 – f)*** is the **fraction of the time spent in the parallel part**:

$$f = \frac{C(seq)}{C(seq) + \frac{C(par)}{p}} \qquad and \qquad (1-f) = \frac{\frac{C(par)}{p}}{C(seq) + \frac{C(par)}{p}}$$

Gustafson-Barsis' Law

Then:

$$C(seq) = f \times \left(C(seq) + \frac{C(par)}{p} \right)$$

$$C(par) = p \times (1-f) \times \left(C(seq) + \frac{C(par)}{p} \right)$$

Simplifying:

$$S(p) \leq \frac{(f + p \times (1-f)) \times \left(C(seq) + \frac{C(par)}{p} \right)}{C(seq) + \frac{C(par)}{p}}$$

$$\Rightarrow \quad S(p) \leq f + p \times (1-f) \qquad \Rightarrow \qquad S(p) \leq p + f \times (1-p)$$

Gustafson-Barsis' Law

Let ***$0 \leq f \leq 1$*** be the fraction of parallel computation spent executing sequential computations then Gustafson-Barsis' law tells us that the maximum speedup that a parallel application can achieve with ***p*** processing units is:

$$S(p) \leq p + f \times (1 - p)$$

While Amdahl's law **starts from the sequential computation** to estimate the maximum speedup that can be achieved with multiple processing units, Gustafson-Barsis' law **starts from a parallel computation** to estimate the maximum speedup that can be achieved for that particular parallel computation.

Gustafson-Barsis' Law

Consider that a certain application executes in 220 seconds in 64 processing units. What is the maximum speedup of the application for 64 processing units knowing that, by experimentation, 5% of the execution time is spent on sequential computations.

$$S(p) \leq 64 + (0.05) \times (1 - 64)$$

$$S(p) \leq 64 - 3.15$$

$$S(p) \leq 60.85$$

Gustafson-Barsis' Law

Consider that a certain company wants to buy a supercomputer with 16384 processors to achieve a speedup of 15000 in an important and fundamental problem. What is the maximum fraction of the parallel computation that can be spent in sequential computations to achieve the expected speedup?

$$15000 \leq 16384 + f \times (1 - 16384)$$

$$f \times 16383 \leq 1384$$

$$f \leq 0.084$$

Limitations of Gustafson-Barsis' Law

By using the parallel computation as the starting point, instead of the sequential computation, the Gustafson-Barsis law assumes that the execution time with one processing unit is, in the worst case, ***p*** times slower than the execution with ***p*** processing units.

This may not be true if the available **memory for the execution with one processing unit is insufficient** when compared to the computation with ***p*** processing units. For this reason, the estimated speedup by the Gustafson-Barsis law is often designated as ***scaled speedup***.

Karp-Flatt Metric

Let us consider again the definitions of sequential and parallel execution time:

$$T(1) = C(seq) + C(par)$$

$$T(p) = C(seq) + \frac{C(par)}{p} + C(com)$$

Let ***e*** be the **experimentally determined sequential fraction of a parallel computation**:

$$e = \frac{C(seq)}{T(1)}$$

Karp-Flatt Metric

Then:

$$C(seq) = e \times T(1)$$

$$C(par) = (1-e) \times T(1)$$

If one considers that **C(com)** is negligible then:

$$T(p) = e \times T(1) + \frac{(1-e) \times T(1)}{p}$$

On the other hand:

$$S(p) = \frac{T(1)}{T(p)}$$

$$\Rightarrow \quad T(1) = S(p) \times T(p)$$

Karp-Flatt Metric

Simplifying:

$$T(p) = e \times S(p) \times T(p) + \frac{(1-e) \times S(p) \times T(p)}{p}$$

$$\Rightarrow \quad 1 = e \times S(p) + \frac{(1-e) \times S(p)}{p}$$

$$\Rightarrow \quad \frac{1}{S(p)} = e + \frac{(1-e)}{p} \qquad \Rightarrow \qquad \frac{1}{S(p)} = e + \frac{1}{p} - \frac{e}{p}$$

$$\Rightarrow \quad \frac{1}{S(p)} = e \times \left(1 - \frac{1}{p}\right) + \frac{1}{p} \qquad \Rightarrow \qquad e = \frac{\frac{1}{S(p)} - \frac{1}{p}}{1 - \frac{1}{p}}$$

Karp-Flatt Metric

Let $S(p)$ be the speedup of a parallel application with $p > 1$ processing units then the Karp-Flatt metric tells us that the experimentally determined sequential fraction is:

$$e = \frac{\frac{1}{S(p)} - \frac{1}{p}}{1 - \frac{1}{p}}$$

The Karp-Flatt metric is interesting because by neglecting the costs with communication/synchronization operations associated with parallelism, allows us **to determine, a posteriori, the relevance of the *C(com)* component** in the eventual decrease of the application's efficiency.

Karp-Flatt Metric

By definition, the experimentally determined sequential fraction is a constant value that does not depend on the number of processing units.

$$e = \frac{C(seq)}{T(1)}$$

On the other hand, the Karp-Flatt metric is a function of the number of processing units.

$$e = \frac{\frac{1}{S(p)} - \frac{1}{p}}{1 - \frac{1}{p}}$$

Karp-Flatt Metric

Considering that the efficiency of an application is a decreasing function on the number of processing units, the Karp-Flatt metric allows us to determine the relevance of ***C(com)*** component in that decrease.

- If the **values of *e* are constant** as we increase the number of processing units then that means that the ***C(com)*** component is also constant, i.e., **the efficiency decrease is due to the scarce parallelism available in the application**
- If the **values of *e* increase** as we increase the number of processing units then that means that the ***C(com)*** component is also increasing, i.e., **the efficienct decrease is due to the excessive costs associated with the parallel computation** (initialization, communication and/or synchronization costs)

The Karp-Flatt metric allows us to detect sources of inefficiency not considered by the model which assumes that ***p*** processing units execute the parallel part ***p*** times faster than executing with just one unit.

Karp-Flatt Metric

Consider the speedups/efficiency obtained by a parallel application:

	2 CPUs	3 CPUs	4 CPUs	5 CPUs	6 CPUs	7 CPUs	8 CPUs
$S(p)$	1.82	2.50	3.08	3.57	4.00	4.38	4.71
$E(p)$	0.910	0.834	0.770	0.714	0.667	0.626	0.589
e	0.099	0.100	0.100	0.100	0.100	0.100	0.100

What is the main reason for the application to just achieve a speedup of 4.71 with 8 processors?

- Given that ***e*** not increases with the number of processors, the main reason for the small speedup is the scarce parallelism avaiable in the application

Karp-Flatt Metric

Consider the speedups/efficiency obtained by a parallel application:

	2 CPUs	3 CPUs	4 CPUs	5 CPUs	6 CPUs	7 CPUs	8 CPUs
$S(p)$	1.87	2.61	3.23	3.73	4.14	4.46	4.71
$E(p)$	0.935	0.870	0.808	0.746	0.690	0.637	0.589
e	0.070	0.075	0.079	0.085	0.090	0.095	0.100

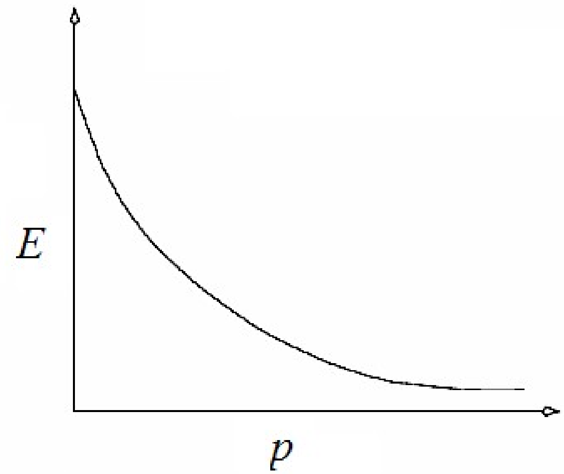
What is the main reason for the application to just achieve a speedup of 4.71 with 8 processors?

- Given that e increases with the number of processors, the main reason for the small speedup are the excessive costs associated to the parallel computation

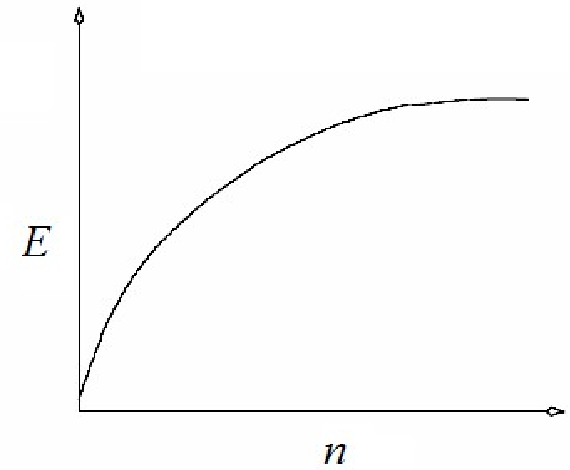
Efficiency and Scalability

Typically, the efficiency of an application is:

- **A decreasing function on the number of processing units**
- **An increasing function on the size of the probem**



Problem size fixed (n)

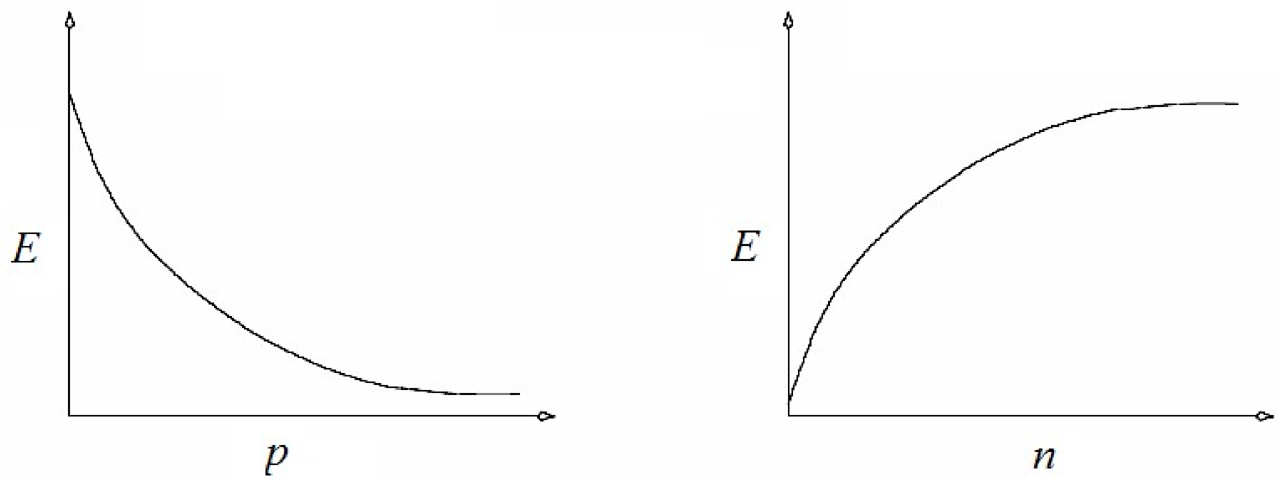


Number of processing units fixed (p)

Efficiency and Scalability

This raises the questions:

- To maintain the same level of efficiency when ***p*** is increased, how should ***n*** be also increased?
- Is the increase in ***n*** sustainable in memory terms, i.e., how does the application scale in terms of memory requirements?



Problem size fixed (n)　　*Number of processing units fixed (p)*

Efficiency and Scalability

An application is said to be **scalable** when it shows capacity to maintain the **same efficiency as the number of processing units and the size of the problem are increased proportionally**.

		1 CPU	2 CPUs	4 CPUs	8 CPUs	16 CPUs
Efficiency	*n* = 10000	1	0.81	0.53	0.28	0.16
	n = 20000	1	0.94	0.80	0.59	0.42
	n = 30000	1	0.96	0.89	0.74	0.58

The scalability of an application reflects its **capacity of making use of more computational resources effectively**.

Isoefficiency Metric

Since the complexity of communication is usually smaller than the complexity of computation, **to maintain the same level of efficiency as we increase the number of processing units one needs to increase the size of the problem**. The isoefficiency metric formalizes this idea.

Let us consider again the definition of speedup:

$$S(p) = \frac{C(seq) + C(par)}{C(seq) + \frac{C(par)}{p} + C(com)} = \frac{p \times \left(C(seq) + C(par)\right)}{p \times C(seq) + C(par) + p \times C(com)}$$

$$= \frac{p \times \left(C(seq) + C(par)\right)}{C(seq) + C(par) + (p-1) \times C(seq) + p \times C(com)}$$

Isoefficiency Metric

Let $T_0(p)$ be the **execution time spent by** p **processing units on the parallel algorithm performing computations not done in the sequential algorithm**:

$$T_0(p) = (p-1) \times C(seq) + p \times C(com)$$

Simplifying:

$$S(p) = \frac{p \times \left(C(seq) + C(par)\right)}{C(seq) + C(par) + T_0(p)}$$

$$E(p) = \frac{C(seq) + C(par)}{C(seq) + C(par) + T_0(p)} = \frac{1}{1 + \dfrac{T_0(p)}{C(seq) + C(par)}} = \frac{1}{1 + \dfrac{T_0(p)}{T(1)}}$$

Isoefficiency Metric

Then:

$$E(p) = \frac{1}{1 + \frac{T_o(p)}{T(1)}}$$

$$\Rightarrow \quad \frac{T_o(p)}{T(1)} = \frac{1 - E(p)}{E(p)} \qquad \Rightarrow \qquad T(1) = \frac{E(p)}{1 - E(p)} \times T_o(p)$$

If one wants to **maintain the same level of efficiency** ***E(p)*** as we increase the number of processing units, then the size of the problem must be increased so that the execution with one unit satisfies:

$$T(1) \geq c \times T_o(p) \qquad where \qquad c = \frac{E(p)}{1 - E(p)}$$

Isoefficiency Metric

Let ***E(p)*** be the efficiency of a parallel application with ***p*** processing units then the isoefficiency metric tells us that, to maintain the same level of efficiency as we increase the number of processing units, the size of the problem must be increased so that the following inequality is satisfied:

$$T(1) \geq c \times T_0(p)$$

$$\text{with} \quad c = \frac{E(p)}{1 - E(p)} \quad \text{and} \quad T_0(p) = (p-1) \times C(seq) + p \times C(com)$$

Isoefficiency Metric

Consider a parallel application where for a problem of size ***n*** we have:

$$T_o(p,n) = n \times p \quad and \quad T(1,n) = 0.1 \times n^2$$

Now, suppose that the desired level of efficiency is ***E(p)=0.9***. Then:

$$T(1,n) \geq \frac{E(p)}{1-E(p)} T_o(p,n)$$

$$0.1 \times n^2 \geq \frac{0.9}{0.1} \times n \times p$$

$$n \geq 90 \times p$$

Meaning that for, say, ***p=10*** then we should have $\boldsymbol{n \geq 900}$.

Or for, say, ***n=2700*** then we should have $\boldsymbol{p \leq 30}$.

Isoefficiency Metric

The isoefficiency metric is an expression of the form $n \geq f(p)$, which establishes conditions to maintain **efficiency regarding execution time, but not memory requirements**. However, the applicability of the isoefficiency metric **depends on the available memory**, since the size of the problem that can be solved is limited by that quantity.

To maintain the same level of efficiency, as we increase the number of processing units, there is a **memory limit on the size of the problem**.

Isoefficiency Metric

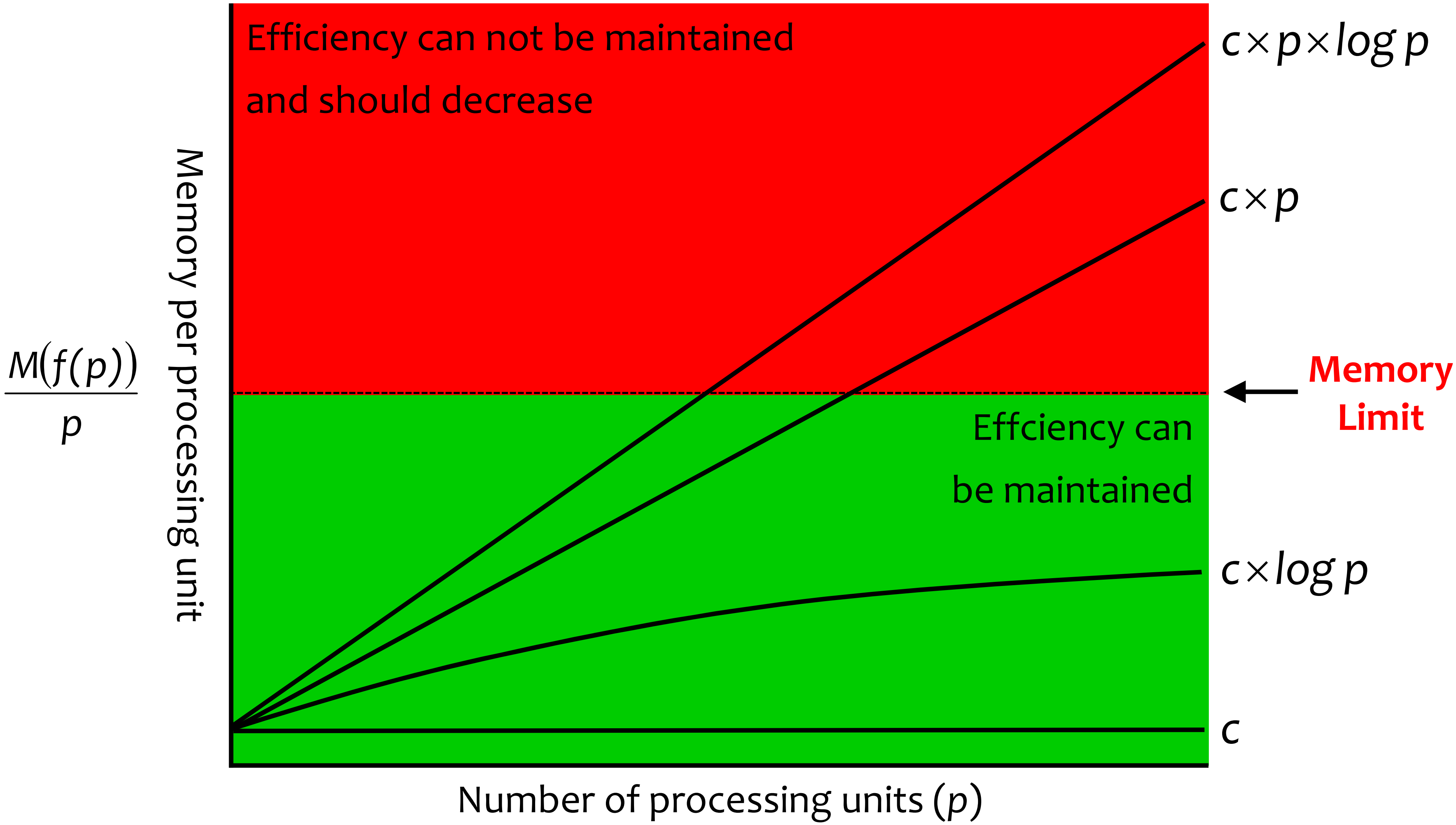
If $M(n)$ specifies the amount of memory required to solve a problem of size n then $M(n)$ **cannot grow arbitrarily**, i.e., beyond the total available memory per processing unit.

We then must have $M(n) \geq M(f(p))$, that is, to maintain the same level of efficiency, the amount of memory required per processing unit is:

$$\frac{M(n)}{p} \geq \frac{M(f(p))}{p}$$

Efficiency cannot be maintained if the amount of memory required per processing unit approximates or exceeds the available memory.

Isoefficiency Metric



Isoefficiency Metric

Consider again the previous example, where:

$$T_o(p,n) = n \times p \quad and \quad T(1,n) = 0.1 \times n^2$$

If ***M(n)=n²*** then:

$$\frac{M(f(p))}{p} = \frac{(90 \times p)^2}{p} = \frac{8100 \times p^2}{p} = 8100 \times p$$

which is an indicator of low scalability.

Otherwise, if ***M(n)=n log(n)*** then:

$$\frac{M(f(p))}{p} = \frac{90 \times p \times log(90 \times p)}{p} = 90 \times log(90 \times p)$$

which is an indicator of better scalability.

Isoefficiency Metric

Consider that the sequential version of a certain application has complexity ***$O(n^3)$*** and that the execution time spent by each of the ***p*** processing units of the parallel version in communication and synchronization operations is ***$O(n^2 \log p)$***. If the amount of memory necessary to represent a problem of size ***n*** is ***n^2***, what is the scalability of the application in terms of memory?

$$n^3 \geq c \times p \times n^2 \times \log p$$

$$n \geq c \times p \times \log p$$

$$M(n) = n^2 \quad \Rightarrow \quad \frac{M(c \times p \times \log p)}{p} = \frac{c^2 \times p^2 \times \log^2 p}{p} = c^2 \times p \times \log^2 p$$

Thus, the scalability of the application is low.